

3+2 neutrinos in a see-saw variation

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If the sterile neutrino mass matrix in an otherwise conventional see-saw model has a rank less than the number of flavors, it is possible to produce pseudo-Dirac neutrinos. For the rank 1 case, 3+2 scenarios devolve naturally as we show by example. Additionally, we find that the lower rank see-saw suppresses some mass differences, so that small mass differences do not require that the individual masses of each neutrino must also be small.

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I. INTRODUCTION

Conventional wisdom holds that neutrinos ought to be Majorana particles with very small masses, due to the action of a “see-saw” mechanism [1], which is built on the concept of quark-lepton symmetry. Alternatively, there have been recent theoretical suggestions [2, 3] that neutrinos may well be Majorana particles occurring in nearly degenerate pairs, the so-called pseudo-Dirac neutrinos. Recent results from Kamiokande [4] on atmospheric neutrinos, from Sudbury [5] on solar neutrinos, and from KamLand [6] on long baseline reactor neutrinos, appear to require oscillations between nearly maximally mixed mass eigenstates. Each of these analyses, however, argues that this mixing cannot be dominantly to sterile states such as are found in pseudo-Dirac pairs. On the other hand, the concatenation of the data from these experiments with that from LSND [7] and other short baseline data does not fit into a theoretical structure which only includes mixing among three active Majorana neutrinos. Many have therefore been motivated to consider the effects of additional (sterile, Majorana) neutrino states, the existence of which is accepted in the conventional “see-saw” extension of the standard model (SM), although there, the actual states are precluded, due to assumed large masses, from appearing directly in experiments.

It should be noted that there is no accepted principle that specifies the flavor space structure of the mass matrix assumed for the sterile sector. Some early discussions [8] implicitly assume that a mass term in the sterile sector should be proportional to the unit matrix. This has the pleasant prospect, in terms of the initial argument for the see-saw, that all neutrino flavors have small masses on the scale of other fermions. However, since there is no obvious requirement that Dirac masses in the neutral lepton sector are the same as Dirac masses in any other fermionic sector, this result is not compelling. Indeed, Goldhaber has argued for a view of family structure and self-energy based masses that naturally produces small neutrino masses [9]. We discuss here a more conventional possibility which arises from a minimal modification of the standard see-saw, namely that the rank of the mass matrix for the sterile sector is less than the number of flavors. Note that this does not conflict with quark-lepton symmetry which applies only to the number and character of states.

In this paper, which is an extension of reference [10], we shall concentrate on the case of a rank 1 sterile matrix, relegating the rank 2 case to some remarks at the end. (The analysis of short baseline data by Sorel, Conrad and Shaevitz [11] suggests that the rank 2 case may

not actually occur in Nature.) We also consider rank 1 to be the more natural case because whatever spontaneous symmetry breaking produces mass in the 3-dimensional sterile flavor space necessarily defines a specific direction. Before including the effects of the sterile mass, we assume three non-degenerate Dirac neutrinos, (although this is not essential,) which are each constructed from one Weyl spinor which is active under the $SU(2)_W$ of the SM and one Weyl spinor which is sterile under that interaction. (Being neutrinos, both Weyl fields have no interactions under the $SU(3)_C$ or the $U(1)$ of the SM.) There is then an MNS [12] matrix which relates these Dirac mass eigenstates to the flavor eigenstates in the usual manner. Note, however, that these matrix elements are not those extracted directly from experiment, as the mass matrix in the sterile sector induces additional mixing.

We next use the Dirac mass eigenstates to define bases in both the 3-dimensional active flavor space and the 3-dimensional sterile flavor space [13]. Following the spirit of the original see-saw, we exclude any initial Majorana mass term in the active space. If the Majorana mass matrix in the sterile space were to vanish also, the three flavors of Dirac neutrinos would be a mixture of (Dirac) mass eigenstates in a structure entirely parallel to that of the quarks.

A rank 1 sterile mass matrix may be represented as a vector of length M oriented in some direction in the 3-dimensional sterile space. If that vector lies along one of the axes, then the Dirac neutrino that would have been formed from it and its active neutrino partner will partake of the usual see-saw structure (one nearly sterile Majorana neutrino with mass approximately M and one nearly purely active neutrino with mass approximately m_D^2/M) and the other two mass eigenstates will remain Dirac neutrinos. If that vector lies in a plane perpendicular to one axis, the eigenstate associated with that axis will remain a pure Dirac neutrino, and the other two will form one pseudo-Dirac pair and a pair displaying the usual see-saw structure. Both of these pairs will be mixtures of the 4 Weyl fields associated with the two mixing Dirac neutrinos. In general, the structure will be 2 pseudo-Dirac pairs and one see-saw pair, all mixed.

As we implied above, the very large mixing required by the atmospheric neutrino measurements could have been taken to be evidence for a scheme involving pseudo-Dirac neutrinos. (This, after all, follows Pontecorvo's initial suggestion [14].) However, pure mixing into the sterile sector is now strongly disfavored [15]. It is evident from the discussion above that there is a region of parameter space (directions of the vector) in which the two pseudo-Dirac

pairs are very nearly degenerate, giving rise to the possibility of strong mixing in the active sector coupled with strong mixing into the sterile sector. We shall explore this point here.

The organization of the remainder of the paper is as follows: In the next section we present the mass matrix, discuss the parameterization of the sterile mass matrix and various limiting cases. We show the spectrum for a general case. In the section following that, we specialize to the case where the vector representing the sterile mass entry lies in a plane perpendicular to one of the axes. In this case we can carry out an analytical expansion in the small parameter $\langle m_D \rangle / M$. In the fourth section we apply those results to the case where the plane in question is perpendicular to the axis for the middle value Dirac mass eigenstate, raising the possibility of near degeneracy between pseudo-Dirac pairs. Moving away from that plane produces large mixing amongst the members of those pseudo-Dirac pairs. Finally, we remark on the structures expected for a rank 2 sterile matrix and then reiterate our conclusions.

II. GENERAL MASS MATRIX

The flavor basis for the active neutrinos and the pairing to sterile components defined by the (generally not diagonal) Dirac mass matrix could be used to specify the basis for the sterile neutrino mass matrix, M_S . Instead we take the basis in the sterile subspace to allow the convention described below. This implies a corresponding transformation of the Dirac mass matrix, which is irrelevant at present since the entries in that matrix are totally unknown.

We now define our convention for the choice of axes in the sterile subspace. Denote the nonzero mass eigenvalue of the rank 1 by M and choose its eigenvector initially in the third direction. Then rotate this vector, first by an angle of θ in the 1 – 3 plane and then by ϕ in the 1 – 2 plane. The rotation is chosen so that the Dirac mass matrix which couples the active and sterile neutrinos becomes diagonal, i.e., the basis is defined by Dirac eigenstates. This produces a 3×3 mass matrix in the sterile sector denoted by

$$M_S = M \begin{bmatrix} \cos^2 \phi \sin^2 \theta & \cos \phi \sin \phi \sin^2 \theta & \cos \phi \sin \theta \cos \theta \\ \cos \phi \sin \phi \sin^2 \theta & \sin^2 \phi \sin^2 \theta & \sin \phi \sin \theta \cos \theta \\ \cos \phi \sin \theta \cos \theta & \sin \phi \sin \theta \cos \theta & \cos^2 \theta \end{bmatrix}. \quad (1)$$

In this representation, the Dirac mass matrix is diagonal by construction

$$m_D = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}. \quad (2)$$

Note that there are special cases. For $\theta = 0$ and any value for ϕ ,

$$M_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M \end{bmatrix}. \quad (3)$$

For $\theta = \pi/2$ and $\phi = 0$,

$$M_S = \begin{bmatrix} M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (4)$$

and, for $\theta = \pi/2$ and $\phi = \pi/2$,

$$M_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (5)$$

These are equivalent under interchanges of the definition of the third axis.

The 6×6 submatrix [16] of the full 12×12 is, in block form,

$$\mathcal{M} = \begin{bmatrix} 0 & m_D \\ m_D & M_S \end{bmatrix}. \quad (6)$$

Since we are ignoring CP violation here, no adjoints or complex conjugations of the mass matrices appear.

Note that, in the chiral representation, the full 12×12 matrix is

$$\begin{bmatrix} 0 & \mathcal{M} \\ \mathcal{M} & 0 \end{bmatrix}. \quad (7)$$

Thus the full set of eigenvalues will be \pm the eigenvalues of \mathcal{M} . Where it matters for some analysis we keep track of the signs of the eigenvalues, however for most results we present positive mass eigenvalues.

After some algebra, we obtain the secular equation

$$\begin{aligned}
0 = & \lambda^6 - M\lambda^5 - (m_1^2 + m_2^2 + m_3^2)\lambda^4 \\
& + M[m_3^2 \sin^2 \theta + m_2^2(\sin^2 \theta \cos^2 \phi + \cos^2 \theta)]\lambda^3 \\
& + (m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2)\lambda^2 \\
& - M(m_1^2 m_2^2 \cos^2 \theta + m_2^2 m_3^2 \cos^2 \phi \sin^2 \theta \\
& + m_3^2 m_1^2 \sin^2 \phi \sin^2 \theta)\lambda \\
& - m_1^2 m_2^2 m_3^2.
\end{aligned} \tag{8}$$

This may be rewritten as

$$\begin{aligned}
0 = & (\lambda^2 - m_1^2)(\lambda^2 - m_2^2)(\lambda^2 - m_3^2) \\
& - \lambda M \left(\lambda^4 - \left[m_3^2 \sin^2 \theta + m_2^2(\sin^2 \theta \cos^2 \phi + \cos^2 \theta) \right. \right. \\
& \left. \left. + m_1^2(\sin^2 \theta \sin^2 \phi + \cos^2 \theta) \right] \lambda^2 \right. \\
& \left. + m_1^2 m_2^2 \cos^2 \theta + m_2^2 m_3^2 \sin^2 \theta \cos^2 \phi \right. \\
& \left. + m_3^2 m_1^2 \sin^2 \theta \sin^2 \phi \right).
\end{aligned} \tag{9}$$

The special cases follow directly. For $\theta = 0$, we find

$$(\lambda^2 - m_1^2)(\lambda^2 - m_2^2)(\lambda^2 - M\lambda - m_3^2) = 0, \tag{10}$$

for $\theta = \pi/2$ and $\phi = 0$

$$(\lambda^2 - m_2^2)(\lambda^2 - m_3^2)(\lambda^2 - M\lambda - m_1^2) = 0, \tag{11}$$

and for $\theta = \pi/2$ and $\phi = \pi/2$

$$(\lambda^2 - m_3^2)(\lambda^2 - m_1^2)(\lambda^2 - M\lambda - m_2^2) = 0. \tag{12}$$

If $m_1^2 = m_2^2 = m_3^2 = m^2$, then we find

$$(\lambda^2 - m^2)^2(\lambda^2 - M\lambda - m^2) = 0. \tag{13}$$

Due to the wide range of possibilities inherent in the system, it is useful to examine specific numerical examples. For the current exercise, we have picked the following parameters:

$$M = 1000$$

$$m_1 = 1$$

$$m_2 = 2$$

$$m_3 = 3.$$

For this choice, the eigenvalues have a definite pattern for all values of θ and ϕ . There are two very close pairs, with mass eigenvalues between 1 and 3. There is one very small eigenvalue, of order 10^{-3} reflecting the ratio of m to M , and one large eigenvalue of order 10^3 (i.e., of order M). Treating the last two as a pair despite their disparity in mass allows us to present results as three tables, one for each pair, for sets of angles $\theta, \phi = \pi/8, \pi/4, 3\pi/8$.

First, for the lower mass close pair, we have

$\theta \backslash \phi$	$\pi/8$	$\pi/4$	$3\pi/8$	
$\pi/8$	1.398125	1.230175	1.068477	
	1.394934	1.228025	1.067688	
$\pi/4$	1.809478	1.478863	1.151936	
	1.808183	1.477134	1.150941	
$3\pi/8$	1.877166	1.562977	1.18999	
	1.876742	1.561911	1.189146	(14)

Then, for the next mass pair with close eigenvalues, we find

$\theta \backslash \phi$	$\pi/8$	$\pi/4$	$3\pi/8$
$\pi/8$	2.038992	2.107688	2.158044
	2.038729	2.107156	2.157407
$\pi/4$	2.347974	2.46348	2.529128
	2.346047	2.462176	2.52809
$3\pi/8$	2.816525	2.847539	2.868607
	2.815691	2.846972	2.868186

(15)

Finally, even though it does not directly impact the argument, we display the remaining pair in order to present a complete set.

$\theta \backslash \phi$	$\pi/8$	$\pi/4$	$3\pi/8$
$\pi/8$	1000.008	1000.008	1000.008
	0.00444	0.005366	0.006778
$\pi/4$	1000.005	1000.006	1000.006
	0.001997	0.002717	0.004248
$3\pi/8$	1000.003	1000.003	1000.004
	0.001289	0.001819	0.003092

(16)

III. TWO FLAVOR SUBSPACE

In the next section we shall discuss the case where two of the pseudo-Dirac pairs are nearly degenerate and follow the mixing patterns as we move away from that region of parameter space. To facilitate that discussion, and to explore a system where analytic approximations are available, we find it useful to examine the limit where one Dirac mass eigenstate remains uncoupled from all of the other states. Anticipating the following section, we decouple m_2 .

This is equivalent to examining a two flavor system in which the Dirac mass eigenvalues are m_1 and m_3 and the vector describing the sterile mass is described by $\phi = 0$.

It is convenient to define some new symbols:

$$m_0^2 = m_1^2 c^2 + m_3^2 s^2 \quad (17)$$

$$a = \frac{(m_1^2 - m_3^2) sc}{m_0 \sqrt{2}} \quad (18)$$

$$b = \frac{m_1 m_3}{m_0} \quad (19)$$

where $c = \cos \theta$, $s = \sin \theta$. Note the additional $1/\sqrt{2}$ factor in the definition of a .

Unitary transformation of the mass matrix

$$\mathcal{M}_{init} = \begin{pmatrix} 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_3 \\ m_1 & 0 & Ms^2 & Mcs \\ 0 & m_3 & Mcs & Mc^2 \end{pmatrix} \quad (20)$$

into the form

$$\mathcal{M}_{fin} = \begin{pmatrix} m_0 & 0 & 0 & a \\ 0 & -m_0 & 0 & -a \\ 0 & 0 & 0 & b \\ a & -a & b & M \end{pmatrix} \quad (21)$$

makes it apparent that, to lowest order, the three small eigenvalues are $\pm m_0, 0$. Note the minus sign on the a in the (2,4) and (4,2) positions.

The matrix, Ω , which effects the transformation, $\mathcal{M}_{fin} = \Omega^\dagger \mathcal{M}_{init} \Omega$, to such a form is

$$\Omega = m_0^{-1} \begin{pmatrix} m_1 c / \sqrt{2} & -m_1 c / \sqrt{2} & m_3 s & 0 \\ -m_3 s / \sqrt{2} & m_3 s / \sqrt{2} & m_1 c & 0 \\ m_0 c / \sqrt{2} & m_0 c / \sqrt{2} & 0 & m_0 s \\ -m_0 s / \sqrt{2} & -m_0 s / \sqrt{2} & 0 & m_0 c \end{pmatrix} \quad (22)$$

This suggests that writing the characteristic equation as

$$\mu (m_0^2 - \mu^2) (M - \mu) = 2\mu^2 a^2 - (m_0^2 - \mu^2) b^2 \quad (23)$$

which is convenient for iterative solution in a series in M^{-1} . The usual form of the secular equation obtained directly from $|\mathcal{M}_{init} - \mu| = 0$, is

$$\mu^4 - \mu^3 M - \mu^2 (m_1^2 + m_3^2) + \mu m_0^2 M + m_1^2 m_3^2 = 0, \quad (24)$$

which is, of course, the same equation.

The solution to order M^{-2} is

$$\mu_1 = m_0 - \frac{a^2}{M} - \frac{a^2}{m_0 M^2} \left(m_0^2 - \frac{a^2}{2} - b^2 \right) \quad (25)$$

$$\mu_2 = -m_0 - \frac{a^2}{M} + \frac{a^2}{m_0 M^2} \left(m_0^2 - \frac{a^2}{2} - b^2 \right) \quad (26)$$

$$\mu_3 = -\frac{b^2}{M} \quad (27)$$

$$\mu_4 = M + \frac{b^2}{M} + 2\frac{a^2}{M} \quad (28)$$

Notice that the eigenvalues sum to M as they must; however, the $\pm m_0$ leading order eigenvalues are shifted in the same direction at $O(M^{-1})$ and in opposite directions at $O(M^{-2})$. The solutions near $\pm m_0$ now specialize to the simpler case discussed in Ref.[2]. The terms at $O(M^{-2})$ effectively only change the value of m_0 . Note that μ_3 and μ_4 , do not acquire $O(M^{-2})$ corrections; their next correction is at the next higher order.

IV. TWO NEARLY DEGENERATE PSUDO-DIRAC PAIRS

Applying the techniques of the last section, we find the angle θ such that m_2 and the eigenvalue for the pseudo-Dirac pair above, m_0 , are approximately degenerate. We then vary ϕ away from 0 and display the eigenfunctions. To illustrate the general nature of the result, we have changed the Dirac masses from the even spacing used above.

In the Table, the Dirac masses are taken to be $m_1 = 1$, $m_2 = 1.1$, and $m_3 = 3$. This effectively means that m_1 is taken to set the pseudo-Dirac mass scale. In order to display the structure of the spectrum, we have chosen $M = 1000$, rather than a larger value, expected to be more realistic, but which would suppress the difference scale between the pairs. The angles are given in degrees.

The Table represents only a small part of the available parameter space and is chosen to display some interesting possible features. First, θ has been chosen so that, at $\phi = 0$, the Dirac pair at m_2 is bracketed by the pseudo-Dirac pair. Such a value of θ exists for any pattern of the Dirac masses. Then, for small values of ϕ , there are always two nearly degenerate pseudo-Dirac pairs.

Note that, for $\phi = 0$, there is no mixing between the field labelled by 2 and the remaining fields, while for the next entry at $\phi = 2.25$ degrees there is considerable mixing. That

mixing increases with ϕ as the difference between the eigenvalues increases. The pattern described by the centroids of the pseudo-Dirac pairs is fixed by the angles θ and ϕ . If M is increased, that pattern hardly changes. The primary effect of increasing M , consistent with the analysis in the previous section, is to decrease the separation of the two members of each pseudo-Dirac pair while producing the usual see-saw behavior for the remaining pair. Thus, tiny differences in mass between masses that are not especially small themselves, are, in the usual sense of the term, natural, in this approach.

The implication for oscillation phenomena is clear. A given weak interaction produces an active flavor eigenstate which is some linear combination of the three active components listed in the Table. That then translates into a linear combination of the six mass eigenstates. From the Table, it is clear that the involvement of the heavy Majorana see-saw state is minimal, so the system effectively consists of the light Majorana see-saw state and the four Majorana states arising from the two pseudo-Dirac pairs. These five states include all three active neutrinos, generating a natural 3+2 scenario.

Since these five mass eigenstates have both active and sterile components, the subsequent time evolution will involve both flavor changing oscillations and oscillation into (and back out of) the sterile sector. This can lead to very complex oscillation patterns, as there are 10 mass differences, 4 of which are independent. For example, a short baseline neutrino oscillation experiment may find that data are better fit by a highly non-sinusoidal function. An example of an appearance probability for the second set of angle parameters in the Table is shown in the Figure.

Finally, inspection of the column labelled “1active” for $\phi = 2.25$ or $\phi = 4.5$, for example, shows that the presence of a rank 1 sterile mass matrix can seriously change any mixing pattern of the MNS type [12], from that which would have obtained with purely Dirac neutrinos.

V. RANK 2

We have not discussed the case of rank 2 matrices explicitly, although the pattern is obvious. In such a case, there would be two see-saw pairs and one pseudo-Dirac pair, leading to three active and one sterile light neutrino. While this pattern has been analyzed in the literature, we do not find any compelling pattern for it in the sterile sector. Furthermore, the

current consistency of all neutrino oscillation data can be accomodated much more easily (and perhaps only, as indicated by Ref.[11],) in the rank 1 case discussed in this paper. Therefore we leave the discussion of rank 2 to another time.

VI. CONCLUSIONS

We have considered here the effects on neutrinos in the SM of the recurrently successful and conventional constraint of quark-lepton symmetry: The existence of six independent Weyl spinor fields of neutrinos, three corresponding to active and three corresponding to sterile neutrinos. In the now venerable see-saw approach, the latter three effectively disappear from the excitation spectrum, leaving small Majorana masses for the active states as a residuum. We have generalized this by considering the effect on the system of a rank less than three character of the 3×3 mass matrix in the sterile sector and studied the rank 1 case, in particular.

In the rank 1 case that we have focused on, we find that the neutrino fields naturally form into two pseudo-Dirac pairs of fields, leaving only one almost pure Majorana active neutrino and one conventionally very heavy sterile Majorana neutrino. We further find that the naturally strong mixing between the active and sterile parts of the two pseudo-Dirac pairs can easily affect the mixing between active neutrinos which is otherwise small, i.e., that due to the Dirac mass matrix induced mixing analogous to what is known to occur in the quark sector, amplifying such small mixing between active neutrinos to much larger values.

We have chosen a limited relative value of the sterile neutrino mass scale, M , that allows for easy discernment of the nature of the effects. It should be noted, however, that the primary effect of increasing M is to decrease the separation of the two members of each pseudo-Dirac pair while producing the usual see-saw behavior for the remaining pair. Thus, tiny differences in mass between masses that are not especially small themselves are, in the usual sense of the term, natural in this approach. This is contrary to the general expectation that the small mass differences responsible for the observed neutrino oscillation phenomena presage small absolute masses for all of the neutrinos.

The above is most easily discerned in the case when the Dirac mass terms for the neutrinos are well separated in value. It remains conceivable that, if their differences are small for

some other reason, then the splitting between the pseudo-Dirac pairs may be larger than that between flavors. In this case, it is still true that large flavor mixing is naturally induced.

VII. ACKNOWLEDGMENTS

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TABLE: Eigenmasses for various values of θ and ϕ for cases of two approximately degenerate pseudo-Dirac pairs.

 $\theta = 9.324078, \phi = 0$

mass	1active	2active	3active	1sterile	2sterile	3sterile
1.099328	0.635032	0.000000	-0.310533	0.698108	0.000000	-0.113793
1.100680	-0.633620	0.000000	0.314383	0.697413	0.000000	-0.115345
1.100000	0.000000	0.707107	0.000000	0.000000	0.707107	0.000000
1.100000	0.000000	-0.707107	0.000000	0.000000	0.707107	0.000000
0.007438	0.441883	0.000000	0.897064	-0.003287	0.000000	-0.002224
1000.008789	0.000162	0.000000	0.002960	0.162017	0.000000	0.986784

$\theta = 9.324078, \phi = 2.25$

mass	1active	2active	3active	1sterile	2sterile	3sterile
1.095953	0.479130	-0.468214	-0.225940	0.525106	-0.466489	-0.082539
1.096608	0.437964	-0.514829	-0.208027	-0.480274	0.513243	0.076041
1.103359	0.416946	0.529767	-0.212981	0.460041	0.531383	-0.078333
1.104056	-0.458049	-0.484588	0.235669	0.505710	0.486376	-0.086730
0.007438	0.441553	0.015769	0.897088	-0.003285	-0.000109	-0.002224
1000.008789	0.000162	0.000007	0.002960	0.161892	0.006361	0.986784

$\theta = 9.324078, \phi = 4.5$

mass	1active	2active	3active	1sterile	2sterile	3sterile
1.092254	0.479875	-0.471453	-0.217491	0.524155	-0.468127	-0.079183
1.092888	-0.458763	0.495815	0.209390	0.501371	-0.492614	-0.076279
1.107010	0.416602	0.526536	-0.221472	0.461189	0.529886	-0.081725
1.107726	0.437718	0.503654	-0.234323	-0.484866	-0.507196	0.086521
0.007439	0.440571	0.031517	0.897156	-0.003273	-0.000217	-0.002226
1000.008789	0.000162	0.000014	0.002960	0.161517	0.012712	0.986784

$$\theta = 9.324078 \quad \phi = 22.5$$

mass	1active	2active	3active	1sterile	2sterile	3sterile
1.062925	0.550356	-0.405921	-0.179528	0.584987	-0.392239	-0.063608
1.063381	0.546548	-0.411257	-0.179609	-0.581185	0.397574	0.063663
1.134871	0.337840	0.568726	-0.249457	0.383405	0.586755	-0.094367
1.135731	0.341702	0.564710	-0.254038	-0.388074	-0.583058	0.096172
0.007475	0.409265	0.154109	0.899298	-0.003058	-0.001048	-0.002241
1000.008789	0.000150	0.000068	0.002960	0.149684	0.062001	0.986784

$$\theta = 9.324078 \quad \phi = 45$$

mass	1active	2active	3active	1sterile	2sterile	3sterile
1.030458	0.632073	-0.290233	-0.127244	0.651329	-0.271878	-0.043708
1.030692	-0.630801	0.292859	0.127989	0.650162	-0.274406	-0.043972
1.163620	0.226485	0.612428	-0.270973	0.263544	0.647849	-0.105102
1.164612	0.227955	0.610618	-0.274587	-0.265472	-0.646488	0.106595
0.007566	0.315141	0.286490	0.904762	-0.002384	-0.001969	-0.002282
1000.008789	0.000115	0.000126	0.002960	0.114563	0.114563	0.986784

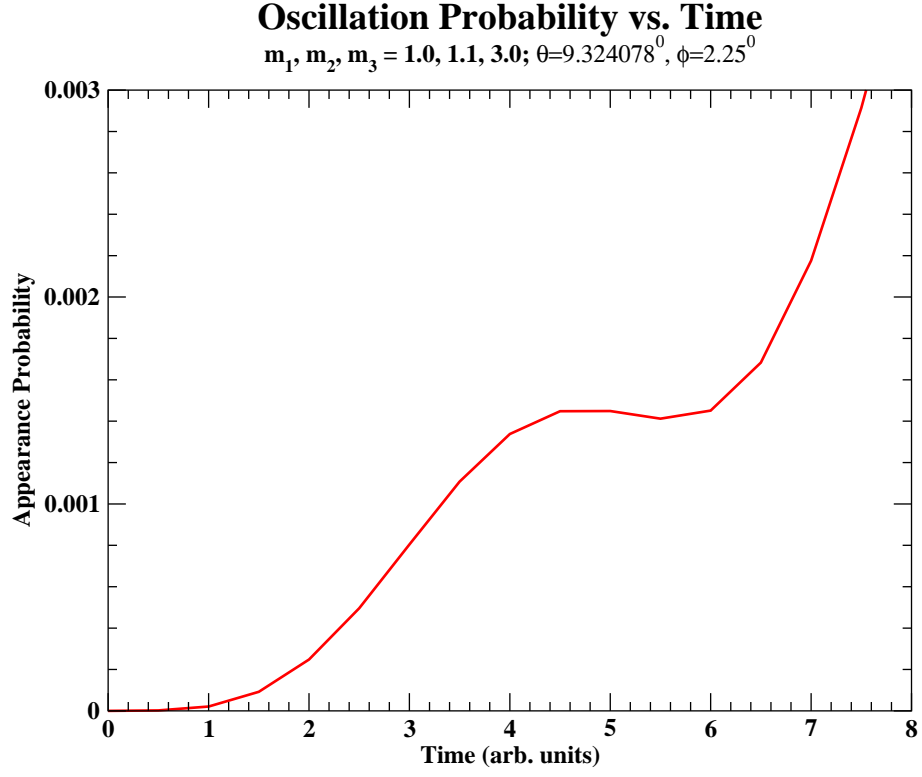


FIG. 1: Appearance probability for one neutrino flavor using the second set of angle parameters in the Table.